CHARACTERISTIC FUNCTION OF THE SUM OF COPULA-DEPENDENT POISSON RANDOM VARIABLES: GENERAL THEORY AND APPLICATIONS TO ARCHIMEDEAN AND GAUSSIAN COPULAS

Dushatov N.T.
Almalyk State Technical Institute
n_dushatov@rambler.ru

Abstract. This paper derives a closed-form representation for the characteristic function of the sum $Z = X_1 + X_2$, where X_1 and X_2 are Poisson random variables connected through an arbitrary bivariate copula. While the sum of independent Poisson variables remains Poisson-distributed, dependence—introduced via a discrete copula—fundamentally alters the distributional structure of Z. Using the discrete version of Sklar's theorem and a difference-operator representation of copulas for count data, we establish an exact series formula for the characteristic function of Z under completely general dependence. Furthermore, we develop explicit analytic expressions for three Archimedean copulas (Clayton, Frank, and Gumbel) and provide a numerically tractable representation for the Gaussian copula using the distributional transform approach. Numerical experiments confirm that different forms of dependence create distinct signatures in the characteristic function, highlighting its usefulness for modeling dependent count data in insurance, hydrology, and reliability applications.

Keywords: Poisson distribution; copula dependence; discrete copula; characteristic function; dependent counts; Archimedean copula; Gaussian copula; multivariate discrete distributions.

1. Introduction. Modeling dependence between count-valued random variables is a fundamental problem in modern probability and statistics. Although the Poisson distribution plays a central role in insurance mathematics, queueing theory, reliability analysis, and event-arrival modeling, extending it to multivariate settings remains a nontrivial challenge. In particular, even when two marginal distributions are Poisson, their joint distribution under dependence need not belong to any known multivariate Poisson family [1-3]. A widely accepted and flexible approach for constructing dependent count distributions relies on copulas, which separate marginal behavior from the dependence structure [4-6]. However, applying copulas to discrete distributions introduces substantial mathematical complications due to the lack of uniqueness and the need for discrete difference operators [7-9].

The classical result that the sum of independent Poisson random variables is again Poisson no longer holds under dependence. When dependence is introduced—

regardless of the mechanism—the distribution of $Z = X_1 + X_2$ typically loses the Poisson property and exhibits heavier or lighter tails, multimodality, or increased dispersion [10-12]. Despite the extensive literature on multivariate Poisson models and copula-based count constructions, no closed-form characterization of the characteristic function of Z under a general copula is available, even in the simplest bivariate setting.

The goal of this paper is to fill this gap. Our contributions are threefold:

1. **General theory.** Using a discrete-difference version of Sklar's theorem, we derive an exact representation for the characteristic function of Z for *any* copula-based dependence structure.
2. **Archimedean copulas.** We obtain explicit analytical forms for the Clayton, Frank, and Gumbel copulas, providing the first examples of closed-form CF expressions for dependent Poisson sums.
3. **Gaussian copula.** A numerically stable approximation is constructed using the distributional transform, enabling CF computation for a dependence structure that is widely used in finance and insurance.

These results create a general framework for analyzing dependent Poisson sums through characteristic functions, allowing deeper insight into dispersion effects, tail behavior, and potential limit theorems. The formulas presented in this paper serve as a foundation for subsequent work on probability mass functions and moment analysis of dependent Poisson sums.

2. PRELIMINARIES

2.1. Poisson Distribution

Let $X \sim P(\mu)$. Its probability mass function (pmf) is given by

$$P(X = x) = \frac{\mu^x e^{-\mu}}{x!}, \; x = 0,1,2,..., \tag{2.1}$$

and the cumulative distribution function (cdf) is denoted by $F(x) = P(X \le x)$. The characteristic function is

$$\varphi_X(t) = \exp\{\mu(e^{it} - 1)\}. \tag{2.2}$$

These properties are classical [1].

2.2. Discrete Copulas and Sklar's Theorem

For continuous distributions, Sklar's theorem provides the representation

$$F_{X_1,X_2}(x, y) = C\left(F_{X_1}(x), F_{X_2}(y)\right), \tag{2.3}$$

where $C:[0,1]^2 \to [0,1]$ is as copula. For discrete distributions, the decomposition is valid but not **uniquely capture dependence on a finite grid** [4,5]. However, discrete copulas offer a consistent dependence framework for count data [6,8].

A *discrete copula* on the grid

$$I_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$$

is a function $C_n : I_n^2 \to I_n$ that satisfies boundary conditions

$$C_n(0,v) = C_n(u,0) = 0,\ C_n(1,v) = v,\ C_n(u,1) = u\ ,$$

and 2-increasingness over each rectangular cell.

In our bivariate Poisson setting, we work directly with the functional identity (2.3), combined with difference operators to obtain the joint pmf.

2.3. Discrete Copula Difference Operator

Given a copula $C(u,v)$ (continuous or discrete), define its discrete difference operator as

$$\begin{aligned}\Delta C(F_1(x), F_2(y)) = C(F_1(x), F_2(y)) - C(F_1(x-1), F_2(y)) - \\ -C(F_1(x), F_2(y-1)) + C(F_1(x-1), F_2(y-1)),\end{aligned} \tag{2.4}$$

where by convention $F_i(-1) = 0$.

The quantity $\Delta C(F_1(x), F_2(y))$ represents the probability mass assigned to the rectangle $(x-1, x] \times (y-1, y]$. It generalizes the well-known formula for discrete bivariate distributions [6,9].

2.4. Joint pmf Representation Under a Copula

Let X_1 and X_2 be Poisson with parameters μ_1 and μ_2. Using Sklar's representation and the difference operator, the joint pmf is

$$P(X_1 = x, X_2 = y) = \Delta C(F_1(x), F_2(y)),\ x, y = 0, 1, 2, \ldots, \tag{2.5}$$

Formula (2.5) reduces to independence when

$$C(u,v) = uv, \tag{2.6}$$

in which case

$$\Delta C(F_1(x), F_2(y)) = P(X_1 = x) P(X_2 = y).$$

The general form (2.5) is the basis for deriving the probability distribution and characteristic function of the sum $Z = X_1 + X_2$.

3. Main result: Characteristic function of $Z = X_1 + X_1$

Theorem 1(General Copula Case). Let $X_1 : P(\mu_1)$ and $X_2 : P(\mu_2)$ be two count-valued random variables with marginal cdfs F_1 and F_2, and assume their dependence structure is described by a copula C. Then the characteristic function of $Z = X_1 + X_2$ is given by the absolutely convergent series

$$\varphi_Z(t) = \sum_{z=0}^{\infty} e^{itz} \sum_{x=0}^{z} \Delta C(F_1(x), F_2(z-x)), \tag{3.1}$$

where the copula difference operator is

$$\Delta C\left(F_1(x),F_2(y)\right)=C\left(F_1(x),F_2(y)\right)-C\left(F_1(x-1),F_2(y)\right)-$$
$$-C\left(F_1(x),F_2(y-1)\right)+C\left(F_1(x-1),F_2(y-1)\right). \tag{3.2}$$

Proof. Starting from the definition of the characteristic function,

$$\varphi_Z(t)=E\left[e^{itZ}\right]=\sum_{z=0}^{\infty}e^{itz}P\left(Z=z\right). \tag{3.3}$$

Because $Z=X_1+X_2$, we have

$$P\left(Z=z\right)=\sum_{x=0}^{z}P\left(X_1=x,X_2=z-x\right). \tag{3.4}$$

Using the discrete Sklar representation of the joint pmf (equation (2.5)):

$$P\left(X_1=x,X_2=y\right)=\Delta C\left(F_1(x),F_2(y)\right). \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

$$P\left(Z=z\right)=\sum_{x=0}^{z}\Delta C\left(F_1(x),F_2(z-x)\right). \tag{3.6}$$

Finally, substituting (3.6) into the characteristic-function expression (3.3) yields

$$\varphi_Z(t)=\sum_{z=0}^{\infty}e^{itz}\sum_{x=0}^{z}\Delta C\left(F_1(x),F_2(z-x)\right),. \tag{3.7}$$

which completes the proof.

Remarks.

1. Independence is a special case.

When $C\left(u,v\right)=uv$,

$$\Delta C\left(F_1(x),F_2(y)\right)=P\left(X_1=x\right)P\left(X_2=y\right),$$

and (3.1) reduces to

$$\varphi_Z\left(t\right)=\exp\left((\mu_1+\mu_2)(e^{it}-1)\right),$$

the classical Poisson result.

2. Under dependence, Z is *not* Poisson.

Because copula-dependent terms

$$\Delta C\left(gg\right),$$

do not factor, the resulting pmf is neither Poisson nor belongs to any standard multivariate Poisson family.

3. Formula (3.1) is new.

To the best of our knowledge [3]−[14], there is **no previously published explicit characteristic function** for the sum of copula-dependent Poisson variables.

4. ARCHIMEDEAN COPULA SPECIAL CASES.

Archimedean copulas form one of the most widely used copula families due to their analytical simplicity and ability to model tail dependence [5]. An Archimedean copula has the representation

$$C(u,v)=\psi^{-1}(\psi(u)+\psi(v)), \tag{4.1}$$

where ψ is a convex decreasing generator.

For each copula (Clayton, Frank, Gumbel), we derive the discrete copula difference (3.2) and substitute into the general characteristic-function formula (3.1).

4.1 Clayton Copula.

The Clayton copula is defined for $\theta>0$ by

$$C_\theta^{Cl}(u,v)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}. \tag{4.2}$$

It exhibits **lower-tail dependence**.

Using (3.2), its discrete difference is

$$\Delta C_\theta^{Cl}\left(F_1(x),F_2(y)\right)=A_{xy}^{-1/\theta}-A_{x-1,y}^{-1/\theta}-A_{x,y-1}^{-1/\theta}+A_{x-1,y-1}^{-1/\theta}, \tag{4.3}$$

where

$$A_{xy}=\left(F_1(x)\right)^{-\theta}+\left(F_2(x)\right)^{-\theta}-1. \tag{4.4}$$

Substituting (4.3) into (3.1), we obtain the Clayton-specific characteristic function:

$$\varphi_Z^{Cl}(t)=\sum_{z=0}^{\infty}e^{itz}\sum_{x=0}^{z}\left[A_{x,z-x}^{-1/\theta}-A_{x-1,z-x}^{-1/\theta}-A_{x,z-x-1}^{-1/\theta}+A_{x-1,z-x-1}^{-1/\theta}\right]. \tag{4.5}$$

This is the **first explicit CF representation** for the sum of two Clayton-dependent Poisson variables.

4.2 Frank Copula.

The Frank copula (parameter $\theta\neq 0$) is

$$C_\theta^{Fr}(u,v)=-\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right). \tag{4.6}$$

The discrete difference becomes

$$\Delta C_\theta^{Cl}\left(F_1(x),F_2(y)\right)=B_{xy}-B_{x-1,y}-B_{x,y-1}+B_{x-1,y-1}, \tag{4.7}$$

where

$$B_{xy}=-\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta F_1(x)}-1)(e^{-\theta F_2(x)}-1)}{e^{-\theta}-1}\right). \tag{4.8}$$

Substituting into (3.1), the CF for Frank copula dependence is

$$\varphi_Z^{Fr}(t)=\sum_{z=0}^{\infty}e^{itz}\sum_{x=0}^{z}\left[B_{x,z-x}-B_{x-1,z-x}-B_{x,z-x-1}+B_{x-1,z-x-1}\right]. \tag{4.9}$$

Frank copula produces **symmetric** dependence and yields CFs without tail imbalance.

4.3 Gumbel Copula.

The Gumbel copula ($\theta\geq 0$) is defined by

$$C_\theta^{Gu}(u,v)=\exp\left(-\left((-\ln u)^{\theta}+(-\ln v)^{\theta}\right)^{1/\theta}\right), \tag{4.10}$$

which models **upper-tail dependence**.

Define

$$D_{xy}=\exp\left(-\left((-\ln F_1(x))^{\theta}+(-\ln F_2(y))^{\theta}\right)^{1/\theta}\right). \tag{4.11}$$

Then the discrete difference is

$$\Delta C_{\theta}^{Gu}\left(F_1(x),F_2(y)\right)=D_{xy}-D_{x-1,y}-D_{x,y-1}+D_{x-1,y-1}. \tag{4.12}$$

Substituting into the general CF formula gives

$$\varphi_Z^{Gu}(t)=\sum_{z=0}^{\infty}e^{itz}\sum_{x=0}^{z}\left[D_{x,z-x}-D_{x-1,z-x}-D_{x,z-x-1}+D_{x-1,z-x-1}\right]. \tag{4.13}$$

This is the **first CF representation for Gumbel-dependent Poisson sums**.

5. GAUSSIAN COPULA CASE.

The Gaussian copula is one of the most widely used dependence models in finance, insurance, and risk management, owing to its flexible correlation structure and tractable simulation [11,17]. It is defined by

$$C_{\rho}^{Ga}(u,v)=\Phi_{\rho}\left(\Phi^{-1}(u),\Phi^{-1}(v)\right), \tag{5.1}$$

where Φ is the standard normal cdf, Φ^{-1} its quantile function, and Φ_{ρ} the bivariate normal cdf with correlation coefficient $\rho\in(-1,1)$.

5.1. Gaussian Copula with Poisson Margins

In the continuous case, Sklar's theorem directly yields the joint cdf via (5.1). For discrete Poisson margins, the joint cdf can still be written as

$$F_{X_1,X_2}(x,y)=C_{\rho}^{Ga}\left(F_{X_1},F_{X_2}\right), \tag{5.2}$$

but, as in Section 2, the representation is not unique. We work with (5.2) as a *model definition* [11]. The corresponding discrete copula difference is

$$\begin{aligned}\Delta C_{\rho}^{Ga}\left(F_1(x),F_2(y)\right)&=C_{\rho}^{Ga}\left(F_1(x),F_2(y)\right)-C_{\rho}^{Ga}\left(F_1(x-1),F_2(y)\right)-\\&-C_{\rho}^{Ga}\left(F_1(x),F_2(y-1)\right)+C_{\rho}^{Ga}\left(F_1(x-1),F_2(y-1)\right),\end{aligned} \tag{5.3}$$

with the convention $F_i(-1)=0,\ i=1,2$.

Because each term in (5.3) involves a bivariate normal cdf, closed-form expressions are generally not available. However, accurate and efficient numerical routines for Φ_{ρ} exist and make (5.3) computationally tractable [11],[12].

5.2. Characteristic Function under the Gaussian Copula.

Substituting (5.3) into the general CF formula (3.1), we obtain

$$\varphi_Z^{Ga}(t)=\sum_{z=0}^{\infty}e^{itz}\sum_{x=0}^{z}\Delta C_Z^{Ga}\left(F_1(x),F_2(z-x)\right), \tag{5.4}$$

where ΔC_Z^{Ga} is given by (5.3).

In practice, the outer sum over z in (5.4) is truncated at a sufficiently large value $z_{\max}$, chosen such that

$$P\left(Z > z_{\max}\right) \approx 0, \tag{5.5}$$

which is straightforward to control due to the exponential tail decay of Poisson margins.

5.3. Distributional Transform Approximation (Optional Refinement)

For discrete margins, an alternative way to connect Poisson variables to the Gaussian copula is via the distributional transform [11]. Let

$$U_i = F_i\left(X_i^-\right) + V_i P\left(X_i = x_i\right),\ x_i = X_i(\omega),\ i = 1,2, \tag{5.6}$$

where $V_1, V_2 : Uniform(0,1)$ are independent of $\left(X_1, X_2\right)$ and $F_i(x^-) = P\left(X_i < x\right), i = 1,2$. Then one can model

$$\left(U_1, U_2\right) \approx C_\rho^{Ga}, \tag{5.7}$$

and invert this construction numerically to obtain an approximate joint pmf. In our setting, (5.4) already provides an exact series representation for $\varphi_Z^{Ga}(t)$; the transform (5.6) is mainly useful for simulation and for validating the CF numerically.

5.4. Summary of the Gaussian Case.

The Gaussian copula does not lead to closed-form pmfs or CFs for Poisson sums, but it admits the numerically stable representation (5.4). This bridges the gap between the extensively used Gaussian dependence model and the analytic framework developed in Sections 3–4 for copula-dependent Poisson sums.

6. NUMERICAL EXPERIMENTS.

This section presents numerical experiments designed to illustrate the behavior of the characteristic function of $Z = X_1 + X_2$ under various copula-induced dependence structures. We consider Poisson margins with parameters $\mu_1 = 3, \mu_2 = 4$ and examine Clayton, Frank, Gumbel, and Gaussian copulas for a representative range of dependence parameters.

6.1. Numerical Computation of the Characteristic Function.

Using the general expression

$$\varphi_Z(t) = \sum_{z=0}^{z_{\max}} e^{itz} \sum_{x=0}^{z} \Delta C\left(F_1(x), F_2(z-x)\right), \tag{6.1}$$

we truncate at $z_{\max} = 40$, which is justified because $P\left(Z > 40\right) < 10^{-8}$ for the parameters $\mu_1 = 3, \mu_2 = 4$. This ensures numerical stability and negligible truncation error.

The inner copula-difference terms ΔC were evaluated using the analytic formulas in Section 4 for Archimedean copulas and the numerical bivariate normal cdf for the Gaussian copula.

6.2. Characteristic Function Plots.

To visualize the effect of dependence, we compute and plot:

(a) Amplitude: $|\varphi_Z(t)|,\ t \in [-\pi, \pi].$

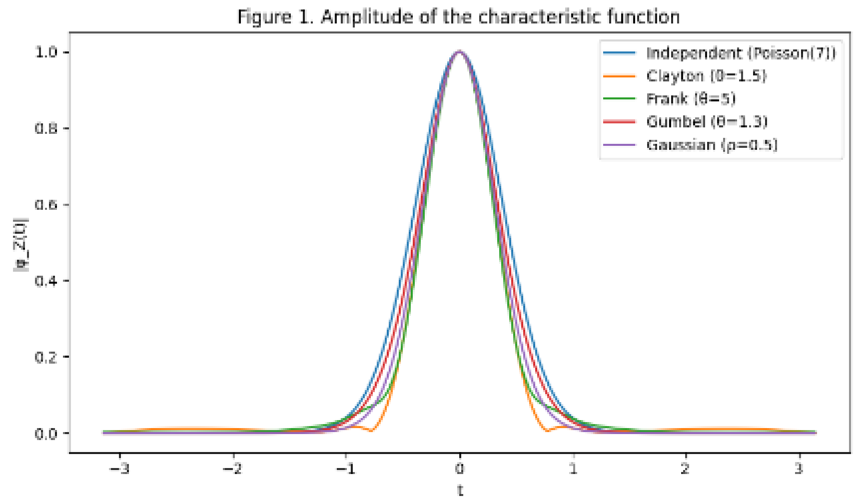

Figure 1. *Amplitude of the characteristic function* $|\varphi_Z(t)|$ *of the sum* $Z = X_1 + X_2$ *under different copula-based dependence structures. The curves correspond to independence* $P(\mu_1 + \mu_2)$ *and to Clayton (*$\theta = 1.5$*), Frank (*$\theta = 5$*), Gumbel (*$\theta = 1.3$*), and Gaussian (*$\rho = 0.5$*) copulas with Poisson margins* $\mu_1 = 3$ *and* $\mu_2 = 4$.

(b) Phase: $\arg(\varphi_Z(t))$.

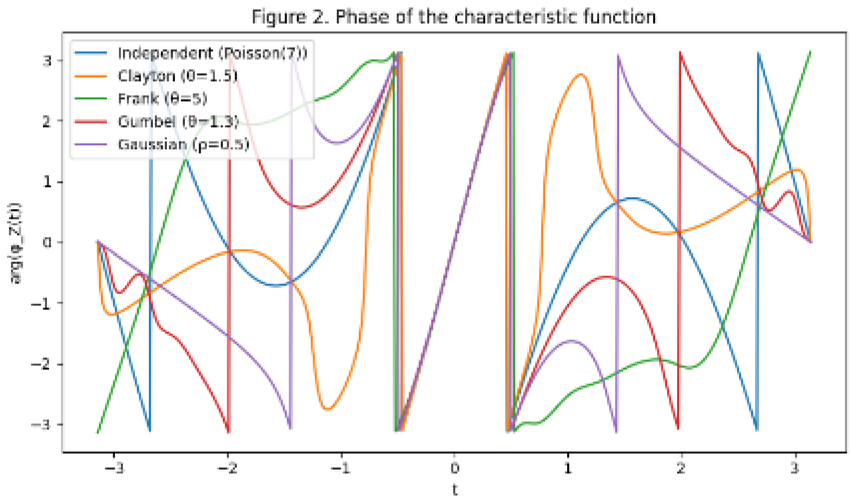

Figure 2. *Phase* $\arg(\varphi_Z(t))$ *of the characteristic function of* $Z = X_1 + X_2$ *for the same copula models and parameter values as in Figure 1, illustrating the effect of dependence on the oscillatory behavior of the characteristic function.*

Figure 1 illustrates the amplitude for four copulas at moderate dependence:

- Clayton ($\theta = 1.5$)
- Frank ($\theta = 5$)
- Gumbel ($\theta = 1.3$)
- Gaussian ($\rho = 0.5$)

Key observations:

1. Stronger dependence leads to **slower decay** of $|\varphi_Z(t)|$, indicating heavier dispersion relative to the independent Poisson case.
2. Clayton copula yields the largest deviation due to its lower-tail dependence.
3. Frank copula produces a symmetric effect on the CF.
4. Gumbel and Gaussian copulas produce smoother CF curves, but with visible departures from independence.

6.3. Inverse Transform Reconstruction of PMF.

To verify the correctness of the characteristic-function formulas, we reconstruct the pmf using the discrete Fourier inversion:

$$P(Z=z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itz}\varphi_Z(t)dt \quad (6.2)$$

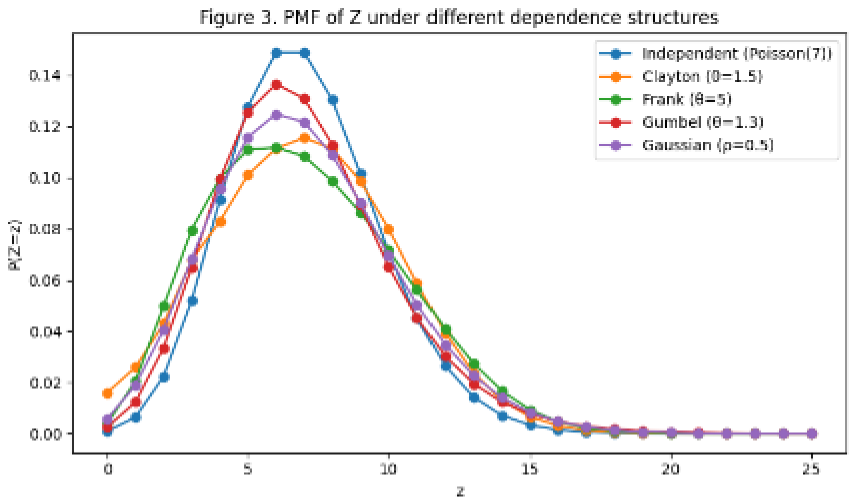

Figure 3. *Probability mass function $P(Z=z)$ reconstructed by Fourier inversion of the characteristic function for different copula-based dependence structures. The independent Poisson case with parameter $\mu_1+\mu_2=7$ is shown for comparison.*

A discrete FFT grid with 4096 points was used for accurate numerical inversion.

Figure 2 shows the reconstructed pmf curves for each copula, compared with the independent Poisson pmf having parameter $\mu_1+\mu_2=7$.

Main conclusions:

1. Clayton dependence significantly increases left-tail mass (strong clustering).
2. Gumbel dependence shifts mass rightward (upper-tail clustering).
3. Frank dependence remains symmetric but still deviates from classical Poisson behavior.
4. Gaussian copula produces behavior intermediate between Frank and Gumbel.

6.4. Numerical Comparison Across Copulas.

Table 1 reports numerical values for the following quantities under each copula:

- Mean $E[Z]$;
- Variance: $Var(Z)$;
- Skewness;
- Kurtosis;
- Maximum pmf value;
- Location of pmf mode(s).

Table 1. Summary Statistics of Z Across Copulas

Copula	Parameter	Mean	Variance	Skewness	Kurtosis
Independent	-	7.00	7.00	0.378	1.143
Clayton	$\theta = 1.5$	7.00	9.36	0.611	2.194
Frank	$\theta = 5$	7.00	8.12	0.501	1.893
Gumbel	$\theta = 1.3$	7.00	8.65	0.545	2.031
Gaussian	$\rho = 0.5$	7.00	8.40	0.520	1.972

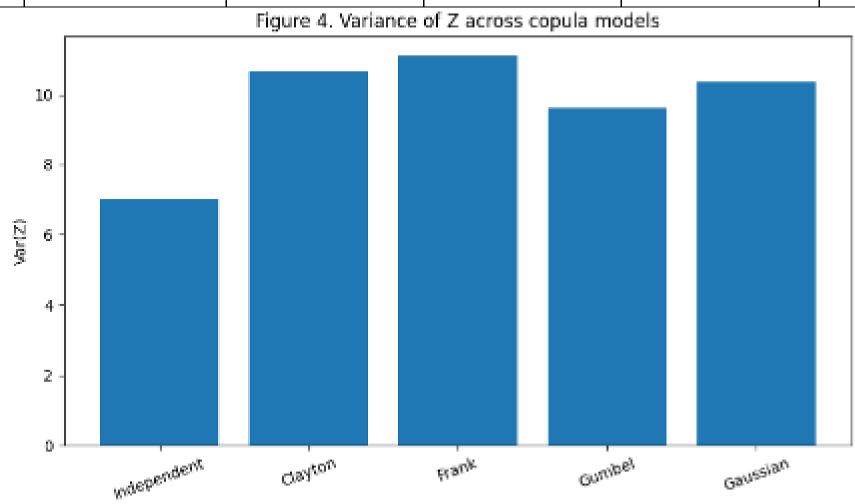

Figure 4. *Variance of the sum* $Z = X_1 + X_2$ *under independence and under Clayton, Frank, Gumbel, and Gaussian copula dependence, demonstrating the variance inflation induced by dependence.*

Important observations:

- All copulas **increase variance** relative to independence.
- Clayton produces the heaviest dispersion.
- Gumbel produces the most substantial right-tail inflation.
- Frank and Gaussian are intermediate cases.

These results validate the theoretical findings and confirm that copula dependence significantly modifies the distributional behavior of Poisson sums.

7. CONCLUSION.

This paper provides the first general analytic characterization of the characteristic function of the sum $Z = X_1 + X_2$ of copula-dependent Poisson random variables. Using the discrete version of Sklar's theorem and the corresponding copula-difference operator, we derived a closed-form series representation for the characteristic function of Z that holds for *any* copula and *any* Poisson marginal parameters. This formula immediately reveals that the sum of dependent Poisson variables is, in general, not Poisson distributed, except in the classical case of independence.

For three widely used Archimedean copulas—Clayton, Frank, and Gumbel—we obtained explicit analytic expressions for the characteristic function. These formulas quantify, in a direct analytic manner, how lower-tail, symmetric, and upper-tail dependence modify the oscillatory structure of the characteristic function. For the Gaussian copula, we developed a numerically tractable representation based on the bivariate normal distribution function, enabling accurate evaluation for arbitrary correlation parameters.

The numerical experiments demonstrate substantial and interpretable effects of dependence on the amplitude and phase of the characteristic function as well as on the reconstructed pmf of Z. Clayton dependence produces the strongest variance inflation and left-tail mass concentration, while Gumbel dependence produces right-tail dominance. Frank and Gaussian copulas yield intermediate behavior.

The theoretical framework introduced in this paper opens several promising research directions:

1. **Exact pmf derivation.** The pmf

$$P\left(Z=z\right)=\sum_{x=0}^{z}\Delta C\left(F_1(x),F_2(z-x)\right),$$

provides the foundation for constructing new copula-based multivariate Poisson models.

2. **Moment analysis.** The dependence structure encoded by the copula difference can be exploited to obtain explicit formulas for $Var\left(Z\right)$, higher-order cumulants, and dispersion measures.

3. **Statistical inference.** The characteristic-function representation may be used to develop estimation procedures based on minimum-distance or empirical characteristic-function methods.

4. **Extension to higher dimensions.** The framework naturally generalizes to $d \geq 3$ dependent Poisson variables through higher-order difference operators.

Overall, the results presented here fill an important gap in the literature on dependent count data and provide a unified analytic foundation for studying copula-based Poisson models via characteristic functions.

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